

Deforming Einstein's Gravity

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ABSTRACT

A deformation of Einstein Gravity is constructed based on gauging the non-commutative $ISO(3,1)$ group using the Seiberg-Witten map. The transformation of the star product under diffeomorphism is given, and the action is determined to second order in the deformation parameter.

1 Introduction

Open string theories as well as D-branes in the presence of background antisymmetric B -field give rise to noncommutative effective field theories [1]-[7]. This is equivalent to field theories deformed with the star product [8],[9]. The primary example of this is noncommutative $U(N)$ Yang-Mills theory [1],[7].

In recent works [10],[11], it was argued that the gravitational field gets deformed and becomes complex [12]. The hermitian metric [13], includes both the symmetric metric and an antisymmetric tensor. The analysis of the linearized theory showed that the theory is consistent, however, further work is needed to show that this is maintained at the non-linear level, a basic problem faced by all theories of nonsymmetric gravity [14]-[16]. For this to happen, it is essential that the diffeomorphism invariance of the real theory is generalized to the complex case. This can happen when both the diffeomorphism transformations and the abelian gauge transformations of the antisymmetric tensor combine to form complex diffeomorphism. The need for this is that gauge symmetry prevents the ghost degrees of freedom present in the antisymmetric tensor from propagating.

The main argument for considering complex vielbein and gauged $U(1, D-1)$ is that for noncommutative Yang-Mills theory it is only possible to gauge the $U(N)$ Lie algebras [7]. Reality conditions necessary to consider $SO(N)$ or $SP(N)$ Lie algebras are not possible. This obstacle was overcome [10],[11], by realizing that it is possible to define subgroups of orthogonal and symplectic subalgebras of noncommutative unitary gauge transformations even though the gauge fields are not valued in the subalgebras of the $U(N)$ Lie algebra. What makes this possible is that one can generalize the reality condition to act on the deformation parameter. Thus the gauge fields are taken to be functions of the deformation parameters θ and the expansion in terms of the non deformed fields is given by the Seiberg-Witten map. To construct a noncommutative gravitational action in four dimensions one proceeds as follows. First the gauge field strength of the noncommutative gauge group $SO(4, 1)$ is taken. This is followed by an Inönü-Wigner contraction to the group $ISO(3, 1)$, thus determining the dependence of the deformed vierbein on the undeformed one. At this stage the construction of the deformed curvature scalar becomes straightforward. The deformed action is computed up to second order in θ . The plan of this paper is as follows. In section two we review the conditions allowing to

deal with noncommutative $SO(N)$ algebras, derive the noncommutative gauge field strengths, perform the group contraction and give the deformed curvature scalar. In section three we expand the action to second order in θ . Section four is the conclusion.

2 Noncommutative Gauging of $SO(4,1)$

The $U(N)$ gauge fields are subject to the condition $\hat{A}_\mu^\dagger = -\hat{A}_\mu$. Such condition can be maintained under the gauge transformations

$$\hat{A}_\mu^g = \hat{g} * \hat{A}_\mu * \hat{g}_*^{-1} - \hat{g} * \partial_\mu \hat{g}_*^{-1}$$

where $\hat{g} * \hat{g}_*^{-1} = 1 = \hat{g}_*^{-1} * \hat{g}$. Therefore, we first introduce the gauge fields $\hat{\omega}_\mu^{AB}$, subject to the conditions [10],[11]

$$\begin{aligned} \hat{\omega}_\mu^{AB\dagger}(x, \theta) &= -\hat{\omega}_\mu^{BA}(x, \theta) \\ \hat{\omega}_\mu^{AB}(x, \theta)^r &\equiv \hat{\omega}_\mu^{AB}(x, -\theta) = -\hat{\omega}_\mu^{BA}(x, \theta) \end{aligned}$$

Expanding the gauge fields in powers of θ , we have

$$\hat{\omega}_\mu^{AB}(x, \theta) = \omega_\mu^{AB} - i\theta^{\nu\rho} \omega_{\mu\nu\rho}^{AB} + \dots$$

The above conditions then imply the following

$$\omega_\mu^{AB} = -\omega_\mu^{BA}, \quad \omega_{\mu\nu\rho}^{AB} = \omega_{\mu\nu\rho}^{BA}$$

A basic assumption to be made is that there are no new degrees of freedom introduced by the new fields, and that they are related to the undeformed fields by the Seiberg-Witten map [7]. This is defined by the property

$$\hat{\omega}_\mu^{AB}(\omega) + \delta_{\hat{\lambda}} \hat{\omega}_\mu^{AB}(\omega) = \hat{\omega}_\mu^{AB}(\omega + \delta_\lambda \omega)$$

where $\hat{g} = e^{\hat{\lambda}}$ and the infinitesimal transformation of ω_μ^{AB} is given by

$$\delta_\lambda \omega_\mu^{AB} = \partial_\mu \lambda^{AB} + \omega_\mu^{AC} \lambda^{CB} - \lambda^{AC} \omega_\mu^{CB}$$

and for the deformed field it is

$$\delta_{\hat{\lambda}} \hat{\omega}_\mu^{AB} = \partial_\mu \hat{\lambda}^{AB} + \hat{\omega}_\mu^{AC} * \hat{\lambda}^{CB} - \hat{\lambda}^{AC} * \hat{\omega}_\mu^{CB}$$

To solve this equation we first write

$$\begin{aligned}\widehat{\omega}_\mu^{AB} &= \omega_\mu^{AB} + \omega_\mu'^{AB}(\omega) \\ \widehat{\lambda}^{AB} &= \lambda^{AB} + \lambda'^{AB}(\lambda, \omega)\end{aligned}$$

where $\omega_\mu'^{AB}(\omega)$ and $\lambda'^{AB}(\lambda, \omega)$ are functions of θ , and then substitute into the variational equation to get [7]

$$\begin{aligned}& \omega_\mu'^{AB}(\omega + \delta\omega) - \omega_\mu'^{AB}(\omega) \\ &= \partial_\mu \lambda'^{AB} + \omega_\mu^{AC} \lambda'^{CB} - \lambda'^{AC} \omega_\mu^{CB} + \omega_\mu^{AC} \lambda^{CB} - \lambda^{AC} \omega_\mu'^{CB} \\ & \quad + \frac{i}{2} \theta^{\nu\rho} (\partial_\nu \omega_\mu^{AB} \partial_\rho \lambda^{CB} + \partial_\rho \lambda^{AC} \partial_\nu \omega_\mu^{CB})\end{aligned}$$

This equation is solved, to first order in θ , by

$$\begin{aligned}\widehat{\omega}_\mu^{AB} &= \omega_\mu^{AB} - \frac{i}{4} \theta^{\nu\rho} \{\omega_\nu, \partial_\rho \omega_\mu + R_{\rho\mu}\}^{AB} + O(\theta^2) \\ \widehat{\lambda}^{AB} &= \lambda^{AB} + \frac{i}{4} \theta^{\nu\rho} \{\partial_\nu \lambda, \omega_\rho\}^{AB} + O(\theta^2)\end{aligned}$$

where we have defined the anticommutator $\{\alpha, \beta\}^{AB} \equiv \alpha^{AC} \beta^{CB} + \beta^{AC} \alpha^{CB}$. With this it is possible to derive the differential equation that govern the dependence of the deformed fields on θ to all orders

$$\delta \widehat{\omega}_\mu^{AB}(\theta) = -\frac{i}{4} \theta^{\nu\rho} \left\{ \widehat{\omega}_{\nu,*} \partial_\rho \widehat{\omega}_\mu + \widehat{R}_{\rho\mu} \right\}^{AB}$$

with the products in the anticommutator given by the star product, and where

$$\widehat{R}_{\mu\nu}^{AB} = \partial_\mu \widehat{\omega}_\nu^{AB} - \partial_\nu \widehat{\omega}_\mu^{AB} + \widehat{\omega}_\mu^{AC} * \widehat{\omega}_\nu^{CB} - \widehat{\omega}_\nu^{AC} * \widehat{\omega}_\mu^{CB}$$

We are mainly interested in determining $\widehat{\omega}_\mu^{AB}(\theta)$ to second order in θ . This is due to the fact that the deformed gravitational action is required to be hermitian. The undeformed fields being real, then implies that all odd powers of θ in the action must vanish. The above equation could be solved iteratively, by inserting the solution to first order in θ in the differential equation and integrating it. The second order corrections in θ to $\widehat{\omega}_\mu^{AB}$ are

$$\begin{aligned}& \frac{1}{32} \theta^{\nu\rho} \theta^{\kappa\sigma} \left(\{\omega_\kappa, 2\{R_{\sigma\nu}, R_{\mu\rho}\} - \{\omega_\nu, (D_\rho R_{\sigma\mu} + \partial_\rho R_{\sigma\mu})\} - \partial_\sigma \{\omega_\nu, (\partial_\rho \omega_\mu + R_{\rho\mu})\}\} \right)^{AB} \\ & \quad + [\partial_\nu \omega_\kappa, \partial_\rho (\partial_\sigma \omega_\mu + R_{\sigma\mu})]^{AB} - \{\{\omega_\nu, (\partial_\rho \omega_\kappa + R_{\rho\kappa})\}, (\partial_\sigma \omega_\mu + R_{\sigma\mu})\}^{AB}\end{aligned}$$

One problem remains of how to determine the dependence of the vierbein \widehat{e}_μ^a on the undeformed field as it is not a gauge field. To resolve this problem we adopt

the strategy of considering the field e_μ^a as the gauge field of the translational generator of the inhomogeneous Lorentz group, obtained through the contraction of the group $SO(4, 1)$ to $ISO(3, 1)$. This is done as follows. First define the $SO(4, 1)$ gauge field ω_μ^{AB} with the field strength

$$R_{\mu\nu}^{AB} = \partial_\mu \omega_\nu^{AB} - \partial_\nu \omega_\mu^{AB} + \omega_\mu^{AC} \omega_\nu^{CB} - \omega_\nu^{AC} \omega_\mu^{CB}$$

and let $A = a, 5$. Define $\omega_\mu^{a5} = k e_\mu^a$. Then we have

$$\begin{aligned} R_{\mu\nu}^{ab} &= \partial_\mu \omega_\nu^{ab} - \partial_\nu \omega_\mu^{ab} + \omega_\mu^{ac} \omega_\nu^{cb} - \omega_\nu^{ac} \omega_\mu^{cb} + k^2 (e_\mu^a e_\nu^b - e_\nu^a e_\mu^b) \\ R_{\mu\nu}^{a5} &\equiv k T_{\mu\nu}^a = k (\partial_\mu e_\nu^a - \partial_\nu e_\mu^a + \omega_\mu^{ac} e_\nu^c - \omega_\nu^{ac} e_\mu^c) \end{aligned}$$

The contraction is done by taking the limit $k \rightarrow 0$. By imposing the condition $T_{\mu\nu}^a = 0$ one can solve for ω_μ^{ab} in terms of e_μ^a . We shall adopt a similar strategy in the deformed case. We write $\hat{\omega}_\mu^{a5} = k \hat{e}_\mu^a$ and $\hat{\omega}_\mu^{55} = k \hat{\phi}_\mu$. We shall only impose the condition $T_{\mu\nu}^a = 0$ and not $\hat{T}_{\mu\nu}^a = 0$ because we are not interested in ϕ_μ which will drop out in the limit $k \rightarrow 0$. The result for \hat{e}_μ^a in the limit $k \rightarrow 0$ is

$$\begin{aligned} \hat{e}_\mu^a &= e_\mu^a - \frac{i}{4} \theta^{\nu\rho} (\omega_\nu^{ac} \partial_\rho e_\mu^c + (\partial_\rho \omega_\mu^{ac} + R_{\rho\mu}^{ac}) e_\nu^c) \\ &\quad + \frac{1}{32} \theta^{\nu\rho} \theta^{\kappa\sigma} (2 \{R_{\sigma\nu}, R_{\mu\rho}\}^{ac} e_\kappa^c - \omega_\kappa^{ac} (D_\rho R_{\sigma\mu}^{cd} + \partial_\rho R_{\sigma\mu}^{cd}) e_\nu^d \\ &\quad - \{\omega_\nu, (D_\rho R_{\sigma\mu} + \partial_\rho R_{\sigma\mu})\}^{ad} e_\kappa^d - \partial_\sigma \{\omega_\nu, (\partial_\rho \omega_\mu + R_{\rho\mu})\}^{ac} e_\kappa^c \\ &\quad - \omega_\kappa^{ac} \partial_\sigma (\omega_\nu^{cd} \partial_\rho e_\mu^d + (\partial_\rho \omega_\mu^{cd} + R_{\rho\mu}^{cd}) e_\nu^d) + \partial_\nu \omega_\kappa^{ac} \partial_\rho \partial_\sigma e_\mu^c \\ &\quad - \partial_\rho (\partial_\sigma \omega_\mu^{ac} + R_{\sigma\mu}^{ac}) \partial_\nu e_\kappa^c - \{\omega_\nu, (\partial_\rho \omega_\kappa + R_{\rho\kappa})\}^{ac} \partial_\sigma e_\mu^c \\ &\quad - (\partial_\sigma \omega_\mu^{ac} + R_{\sigma\mu}^{ac}) (\omega_\nu^{cd} \partial_\rho e_\kappa^d + (\partial_\rho \omega_\kappa^{cd} + R_{\rho\kappa}^{cd}) e_\nu^d)) + O(\theta^3) \end{aligned}$$

At this point, it is possible to determine the deformed curvature and use it to calculate the deformed action given by

$$\int d^4x \sqrt{\hat{e}} * \hat{e}_{*a}^\mu * \hat{R}_{\mu\nu}^{ab} * (e_{*b}^\nu)^\dagger * \sqrt{\hat{e}}^\dagger$$

Notice that this action is hermitian including the measure. We have defined $\hat{e} = \det(\hat{e}_\mu^a)$, and the inverse vierbein by

$$\hat{e}_{*a}^\mu * \hat{e}_\mu^b = \delta_a^b$$

This will determine the inverse deformed vierbein as an expansion in θ . By writing

$$\begin{aligned} \hat{e}_\mu^a &= e_\mu^a + i\theta^{\nu\rho} e_{\mu\nu\rho}^a + \theta^{\nu\rho} \theta^{\kappa\sigma} e_{\mu\nu\rho\kappa\sigma}^a + O(\theta^3) \\ \hat{e}_{*a}^\mu &= e_a^\mu + i\theta^{\nu\rho} e_{a\nu\rho}^\mu + \theta^{\nu\rho} \theta^{\kappa\sigma} e_{a\nu\rho\kappa\sigma}^\mu + O(\theta^3) \end{aligned}$$

where $e_{\mu\nu\rho}^a$ and $e_{\mu\nu\rho\kappa\sigma}^a$ can be read from the expansion of \hat{e}_μ^a to second order in θ , one finds that

$$\begin{aligned} e_{a\nu\rho}^\mu &= -e_b^\mu \left(e_a^\kappa e_{\kappa\nu\rho}^b + \frac{1}{2} \partial_\nu e_a^\kappa \partial_\rho e_\kappa^b \right) \\ e_{a\nu\rho\kappa\sigma}^\mu &= -e_b^\mu \left(e_{a\nu\rho}^\alpha e_{\alpha\kappa\sigma}^b + e_a^\alpha e_{\alpha\nu\rho\kappa\sigma}^b - \frac{1}{4} \partial_\nu \partial_\kappa e_a^\alpha \partial_\rho \partial_\sigma e_\alpha^b - \frac{1}{2} \partial_\nu e_a^\alpha \partial_\rho e_{\alpha\kappa\sigma}^b + \frac{1}{2} \partial_\rho e_\alpha^b \partial_\nu e_{\alpha\kappa\sigma}^\alpha \right) \end{aligned}$$

It is legitimate to question the meaning of the star product under general coordinate transformations, and whether this action is diffeomorphism invariant. Afterall, the original definition of the star product assumes that the commutator

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}$$

is constant. However, under diffeomorphism transformations, $\theta^{\mu\nu}$ becomes a function of x , and one has to generalize the definition of the star product to be applicable for a general manifold. The prescription for doing this was given by Kontsevich [17]. The star product is then defined by

$$\begin{aligned} f * g &= fg + \hbar B_1(f, g) + \hbar^2 B_2(f, g) + \dots \\ &= fg + \hbar \alpha^{ab} \partial_a f \partial_b g + \frac{1}{2} \hbar^2 \alpha^{ab} \alpha^{cd} \partial_a \partial_c f \partial_b \partial_d g + \\ &\quad + \frac{1}{3} \hbar^2 \alpha^{as} \partial_s \alpha^{bc} (\partial_a \partial_b f \partial_c \partial_d g + \partial_a \partial_b g \partial_c \partial_d f) + O(\hbar^3) \end{aligned}$$

where

$$\alpha^{ab} = \theta^{\mu\nu} \partial_\mu z^a \partial_\nu z^b + O(\theta^3)$$

Therefore under diffeomorphisms the star product transforms according to

$$\begin{aligned} * &\rightarrow *' \\ f'(\hbar) &= Df(\hbar) \\ f' *' g' &= D(D^{-1} f' * D^{-1} g') \\ D &= 1 + \sum_{i=1} \hbar D_i \end{aligned}$$

In this case [18]

$$D = 1 - \frac{\hbar^2}{4} \theta^{\mu\nu} \theta^{\rho\sigma} \left(\partial_\mu \partial_\rho z^a \partial_\nu \partial_\sigma z^b \partial_a \partial_b + \frac{2}{3} (\partial_\mu \partial_\rho z^a \partial_\nu z^b \partial_\sigma z^c) \partial_a \partial_b \partial_c \right) + O(\hbar^4)$$

It is thus possible to define the star product to be invariant under diffeomorphism transformations..

3 The action to second order in θ

To determine the action to second order in θ , we first write

$$\widehat{R}_{\mu\nu}^{ab} = R_{\mu\nu}^{ab} + i\theta^{\rho\tau} R_{\mu\nu\rho\tau}^{ab} + \theta^{\rho\tau}\theta^{\kappa\sigma} R_{\mu\nu\rho\tau\kappa\sigma}^{ab} + O(\theta^3)$$

where

$$\begin{aligned} R_{\mu\nu\rho\tau}^{ab} &= \partial_\mu \omega_{\nu\rho\tau}^{ab} + \omega_\mu^{ac} \omega_{\nu\rho\tau}^{ac} + \omega_{\mu\rho\tau}^{ac} \omega_\nu^{cb} - \frac{1}{2} \partial_\rho \omega_\mu^{ac} \partial_\tau \omega_\nu^{cb} - \mu \leftrightarrow \nu \\ R_{\mu\nu\rho\tau\kappa\sigma}^{ab} &= \partial_\mu \omega_{\nu\rho\tau\kappa\sigma}^{ab} + \omega_\mu^{ac} \omega_{\nu\rho\tau\kappa\sigma}^{ac} + \omega_{\mu\rho\tau\kappa\sigma}^{ac} \omega_\nu^{cb} - \omega_{\mu\rho\tau}^{ac} \omega_{\nu\kappa\sigma}^{ac} \\ &\quad - \frac{1}{4} \partial_\rho \partial_\kappa \omega_\mu^{ac} \partial_\tau \partial_\sigma \omega_\nu^{cb} - \mu \leftrightarrow \nu \end{aligned}$$

With this we can now expand

$$\begin{aligned} \widehat{e}_{*a}^\mu * \widehat{R}_{\mu\nu}^{ab} * (e_{*b}^\nu)^\dagger &= R \\ &\quad + \theta^{\rho\tau}\theta^{\kappa\sigma} (e_a^\mu R_{\mu\nu\rho\tau\kappa\sigma}^{ab} e_b^\nu + e_{a\rho\tau\kappa\sigma}^\mu R_{\mu\nu}^{ab} e_b^\nu + e_a^\mu R_{\mu\nu}^{ab} e_{b\rho\tau\kappa\sigma}^\nu \\ &\quad - e_{a\rho\tau}^\mu R_{\mu\nu}^{ab} e_{b\kappa\sigma}^\nu - e_{a\rho\tau}^\mu R_{\mu\nu\kappa\sigma}^{ab} e_b^\nu - e_a^\mu R_{\mu\nu\rho\tau}^{ab} e_{b\kappa\sigma}^\nu) + O(\theta^4) \end{aligned}$$

Notice that the odd powers of θ cancel because of the hermiticity of the above expression and reality of the undeformed fields. The expansion of the determinant is

$$\det(\widehat{e}_\mu^a) = 1 + i\theta^{\rho\tau} e_a^\mu e_{\mu\rho\tau}^a + \frac{1}{2} \theta^{\rho\tau}\theta^{\kappa\sigma} (e_{\mu\rho\tau}^a e_{\nu\kappa\sigma}^a e_c^\mu e_c^\nu - e_{\mu\rho\tau}^a e_{\nu\kappa\sigma}^b e_a^\mu e_b^\nu) + O(\theta^3)$$

This completes the action to second order in θ . Of course the actual expression obtained after substituting for the fields $e_{a\rho\tau}^\mu$, $e_{a\rho\tau\kappa\sigma}^\mu$, $\omega_{\mu\rho\tau}^{ab}$ and $\omega_{\mu\rho\tau\kappa\sigma}^{ab}$ is very complicated, and it is not clear whether one can associate a geometric structure with it. One can, however, take this expression and study the deformations to the graviton propagator, which will receive θ^2 corrections. It is an interesting question to study the effect of the deformation on the renormalizability of the theory. This project will have to be handled by using a computer program for algebraic manipulations.

4 Conclusion

In this work we have shown that it is possible to deform Einstein's gravity without introducing new fields. The idea is based on the gauging of the noncommutative gauge group $ISO(3,1)$ and using the Seiberg-Witten map to express the

deformed fields in terms of the undeformed ones. The reality of the undeformed fields and the hermiticity of the action implies that the lowest order correction to Einstein's action is second order in the deformation parameter. This makes the form of the corrections fairly complicated. It is, however, possible to use perturbation theory to determine the modified graviton propagator as well as the vertex operators, to second order in θ . It is an interesting problem to study the renormalizability of the theory and the effects of the deformation parameter on the infrared and ultraviolet divergencies. Performing these calculations will be left for future work.

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